# Chapter 2

# DERIVATIVES, DIFFERENTIALS, AND DIFFERENTIAL EQUATIONS

**Note:** THIS CHAPTER IS FOR REFERENCE ONLY. If you are feeling insecure about your understanding of Calculus, read Chapters 2 and 3 closely. Otherwise, skim them. In either case, don't panic if the material looks obscure. We'll ease into it gently (besides, the first test will primarily be over Chapter 1).

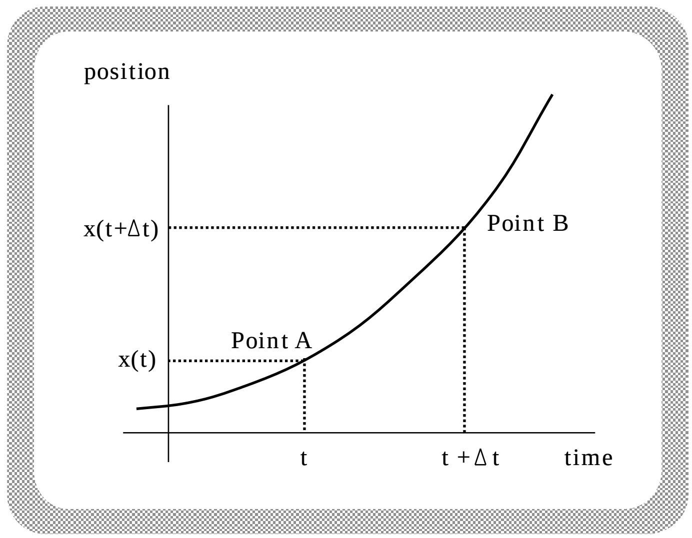


FIGURE 2.1

## A.) Derivatives–Preliminaries:

**1.)** An eccentric's pet ant is constrained to move in one dimension. Figure 2.1 presents a graph of its motion in time.

**a.)** At time *t*, the ant is located at *Point A*. While there, its position coordinate is *x(t)*. At time $t + \Delta t$, the ant is located at *Point B* with a position coordinate $x(t + \Delta t)$.

**b.)** In Figure 2.2, a *secant* is drawn between *Points A* and *B*.

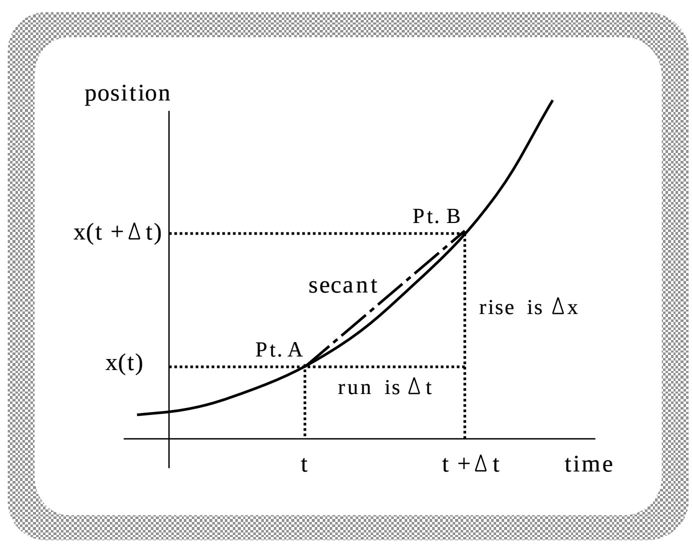


FIGURE 2.2

**i.)** Note that the *slope of the secant* is equal to the secant's "rise over run," or:

$$\text{slope} = \frac{x(t+\Delta t) - x(t)}{(t+\Delta t) - t}$$
$$= \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

**2.)** If we hold *Point A* fixed while allowing $\Delta t$ to become very small, *Point B* approaches *Point A* and the *secant* approaches the *tangent to the curve* at *Point A*. Writing this out mathematically, we get:

$$\text{slope of tangent} = \text{slope of } \mathbf{secant}$$
$$= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

**Note:** Although the denominator goes to zero, the *ratio* of the numerator and denominator converges to a finite number.

**a.)** This operation is called a *derivative*. In general, a *derivative* yields a new function that defines the *rate of change of the original function with respect to one of its variables*. In the above case, it defines the *rate of change of "x" with time*.

**b.)** Books from different disciplines denote *derivative* operations in different ways.

**i.)** Some math books, for instance, use the notation:

$$x'(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

**ii.)** Most physics books, on the other hand, use the notation:

$$\frac{d(x)}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

**Note:** In the latter notation, *dt* is NOT *d* multiplied by a time *t* any more than $\Delta t$ is $\Delta$ multiplied by a time *t*.

**c.)** Mathematicians treat *dx/dt* as a <u>single symbol</u> denoting a particular mathematical operation (a derivative). As you will see shortly, physicists are not so formal in their treatment of the notation.

**3.)** <u>Bottom line</u>: In general, the *time derivative* of a time-varying function *f(t)* gives you a second function *df(t)/dt* that defines the *slope of the tangent to f(t)* at any point on *f(t)'s* curve. Put another way, it gives us a general function that defines *the rate at which f(t) changes with time.* This function can then be evaluated at any given time-of-interest *t*.

## B.) Derivatives–In the Beginning . . . :

**1.)** In most beginning Calculus classes, the *definition of the derivative*:

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t},$$

is used to generate derivative-type expressions for relatively common functions. Although the approach is limited, it is effective in presenting the theoretical underpinnings of the derivative. As such, we will do a few examples to show how it works.

**a.)** Consider the function *f(t) = 3t +2* (see Figure 2.3). What is the *time rate of change of the function*? That is, what is the *new function* that defines how *f(t)* changes as *t* changes?

**i.)** This is an admittedly easy problem. Seeing that the function is linear and noting that its slope (i.e., its rate of change) is constant and equal to 3, we know the answer to the derivative calculation before we start. Nevertheless:

**ii.)** Following through with the *definition of the derivative*:

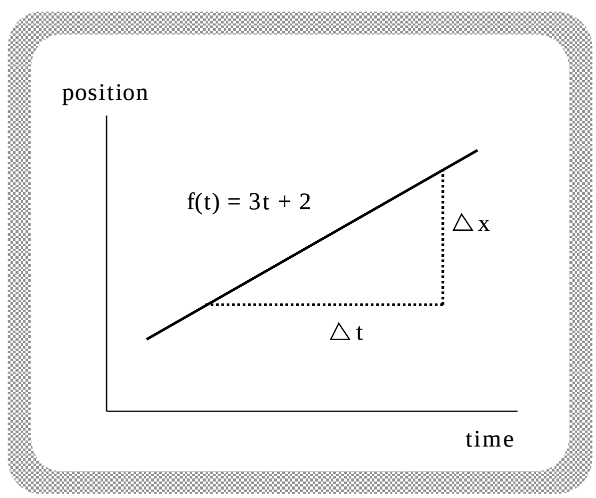


FIGURE 2.3

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{[3(t+\Delta t)+2] - [3t+2]}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{3t + 3\Delta t + 2 - 3t - 2}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{3\Delta t}{\Delta t}$$
$$= 3.$$

**iii.)** As the $\Delta t$*'s* in the fourth line canceled, we didn't need to invoke the limit (i.e., have $\Delta t$ go to zero). Still, the process leads us to a function (a constant in this case) that defines the *rate* at which $x = 3t + 2$ changes with time.

**b.)** Consider the function $x(t) = kt^3$ (see Figure 2.4 below), where $k$ is a proportionality constant equal to $1\ m/s^3$ (without $k$, $x$ would have the units of *seconds cubed*). What is the *rate of change* of the function? That is, what is the *new function* that defines how $x$ changes as $t$ changes?

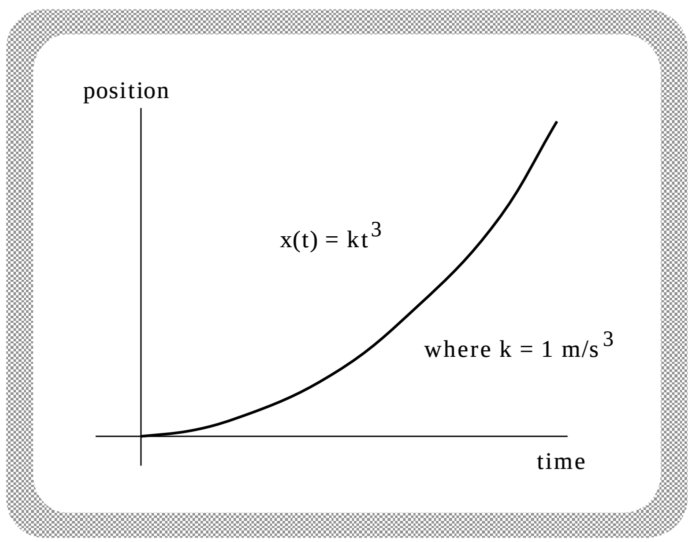

FIGURE 2.4

**i.)** This is not a trivial situation. Setting $k = 1$ (we don't need to carry it through the calculation as it was included only to make the units acceptable), and following through with the *definition of the derivative*, we get:

$$\begin{aligned}
\frac{dx(t)}{dt} &= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\left[(t+\Delta t)^3\right] - \left[t^3\right]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{t^3 + 3t^2(\Delta t) + 3t(\Delta t)^2 + (\Delta t)^3 - t^3}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{3t^2(\Delta t) + 3t(\Delta t)^2 + (\Delta t)^3}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left[3t^2 + 3t(\Delta t) + (\Delta t)^2\right] \\
&= 3t^2 \qquad \text{(or } 3kt^2 \text{ if we replace the "k").}
\end{aligned}$$

**ii.)** What happened to all the *$\Delta t$'s*? Some were canceled; some went away when we invoked the limit.

**iii.)** What does this mean? At *t = 2 seconds*, for instance, the body is at $x = kt^3 = (1\ m/s^3)(2\ s)^3 = 8$ *meters*. In addition, according to our derivation, the *slope of the tangent to the curve* at that time (i.e., the body's *rate of change of* *position* *with time*, a.k.a. its *velocity*) is $dx/dt = 3kt^2 = 3(1\ m/s^3)(2\ s)^2 = 12\ m/s$.

**c.)** If we had done this process for $k_2t^2$, we would have found the derivative equal to $2k_2t$ (try it!). If we had done the process for $k_4t^4$, the derivative would have been $4k_4t^3$.

Generalizing, we can write:

If $x = k_1t$, then $dx/dt = 1k_1$.

If $x = k_2t^2$, then $dx/dt = 2k_2t$.

If $x = k_3t^3$, then $dx/dt = 3k_3t^2$.

If $x = k_4t^4$, then $dx/dt = 4k_4t^3$.

etc.

From this we can deduce that if:

$$x = k_n t^n, \qquad \text{then} \qquad dx/dt = nk_n t^{n-1}.$$

**d.)** This is exactly what Calculus students do for the first month or so of the *Derivatives* section of their course. They use the *definition of the derivative* to determine specific derivative functions, then they generalize if possible.

**e.)** Other derivative rules that might come in handy:

**i.)** $d(Ae^{-kt})/dt = -Ake^{-kt}$, where A and k are constants;

**ii.)** $d[\sin(kt)]/dt = k\cos(kt)$;

**iii.)** $d[\cos(kt)]/dt = -k\sin(kt)$;

**iv.)** $d[\ln(kt)]/dt = k(1/t)$, where "ln" designates a *natural log*.

**v.)** Noting the $(k/t) = kt^{-1}$, we get $d(kt^{-1})/dt = (-1)kt^{-2} = -k/(t^2)$;

**vi.)** $d[f(x)g(x)]/dx = [df(x)/dx][g(x)] + [f(x)][dg(x)/dx]$ (this is called *the product rule*). Example: $d[(4x)(x^2)]/dx = (4x)(2x) + 4(x^2) = 12x^2$ (note that $(4x)(x^2) = 4x^3$, and that the derivative of $4x^3$ is, indeed, $12x^2$ . . . it works).

**C.) The Chasm Between Mathematicians and Physicists:**

**1.)** As has already been mentioned, mathematicians have very definite ideas about how calculus-oriented terms are to be defined and manipulated. On the other hand, physicists are (within limits) willing to play fast and loose with the notation. Specifically:

**a.)** Mathematicians demand that *dt* be read as "an *infinitesimally small* time interval" (i.e., no real duration at all). Physicists think of *dt* as *a very, very small* time interval. The appeal of the latter description? It implies there exists a tiny period of time during which something *can* happen.

**b.)** A similar situation holds for the symbol *dx*. From a mathematician's perspective, *dx* is "an *infinitesimally small* displacement with no *physical* reality to it at all." Physicists treat *dx* as a *very, very small* displacement.

**c.)** Although *dx/dt* denotes a specific operation, physicists are more than happy to treat *dx* and *dt* as separate entities, even to the extent of manipulating them algebraically. For instance, $dx/dt = c$ implies that the derivative of a function *x* equals a constant *c*. Mathematicians use formal algorithms and transforms to show that if this is true, it is also true that $dx = c\,dt$ (this reads *a differential change of position "dx" equals a constant "c" times the differential time interval "dt" over which the displacement takes place*). Physicists know how to use the algorithms, etc., but aren't willing to bother. Instead, they short cut the process by simply multiplying both sides of $dx/dt = c$ by *dt* yielding $dx = c\,dt$. Mathematicians hate this kind of nonchalance, but it works if you're only interested in the bottom line.

**2.)** To what does this all come down? As aesthetically irritating as it might be to mathematicians, the Calculus is both easier to visualize and easier to use in the context of *real world problems* when *dx's* and *dt's* are afforded a physical significance.

**3.)** An example: Understanding *the Chain Rule:*

**a.)** Consider the function $f(x) = x^2$.

**i.)** Note that it is easy to see how the function changes with *changes of position* as its derivative with respect to *x* is:

$$\frac{df(x)}{dx} = 2x.$$

**ii.)** IMPORTANT: There is nothing wrong with manipulating this into the form $df(x) = (2x)dx$, then replacing the *2x* by $df(x)/dx$. This probably seems a bit circular, but in doing so we get an interesting differential equation. Specifically:

$$\begin{aligned} df(x) &= (2x)\ dx \\ &= \left[\frac{df(x)}{dx}\right]dx. \end{aligned}$$

This equation states that the *differential change of the function* (i.e., *df*) as one moves some *differential distance dx* along the *x axis* equals the *rate at which the function changes with "x"* (i.e., $df/dx = 2x$) times the SIZE of the change (i.e., the *displacement* *dx*).

**b.)** Assume now that $x$ changes with time (i.e., $x = x(t)$). Specifically, for example, assume that $x(t) = kt^3$. How does the function $f(x)$ change with time (i.e., what is $df(x)/dt$)?

**c.)** The approach that allows us to determine $df/dt$ is called the Chain Rule. Stated mathematically, it is:

$$\frac{df(x)}{dt} = \left[\frac{df(x)}{dx}\right]\left[\frac{dx(t)}{dt}\right].$$

The question? Where did this come from and what does it mean?

**d.)** In *Part 3a-ii*, we decided that the *change of the function df* as we move some *differential distance dx* along the *x axis* equals the *rate at which the function changes as we proceed along the x axis* (i.e., $df/dx$) times the SIZE of the *differential displacement* ($dx$). Mathematically, this was written:

$$df(x) = \left[\frac{df(x)}{dx}\right]dx \qquad \text{(Equation A).}$$

**e.)** We can follow a similar path in expressing the differential displacement $dx$ in terms of $dt$. That is, the *change* of the position function $dx$ as we move through some *differential time interval dt* equals the *rate at which x changes in time* (i.e., $dx/dt$) times the SIZE of the *differential time interval dt* during which the change occurs. This is written:

$$dx = \left[\frac{dx(t)}{dt}\right]dt \qquad \text{(Equation B).}$$

**f.)** As Equation B is an expression for the *differential change* of $x$ (i.e., $dx$) in terms of time, we can use it in Equation A to write:

$$\begin{aligned} df(x) &= \left[\frac{df(x)}{dx}\right] \quad dx \\ &= \left[\frac{df(x)}{dx}\right]\left[\frac{dx(t)}{dt}\,dt\right]. \end{aligned}$$

**g.)** Dividing both sides by *dt* (oops, there goes Fermat, rolling over in his grave again), we get the Chain Rule, or:

$$\frac{df(x)}{dt}=\left[\frac{df(x)}{dx}\right]\left[\frac{dx(t)}{dt}\right].$$

**h.)** Example: Consider $f(x) = x^2$ (derivative $df(x)/dx = 2x$). If $x(t) = kt^3$ (derivative $dx(t)/dt = 3kt^2$), then $df/dt$ is:

$$\begin{aligned}\frac{df(x)}{dt}&=\left[\frac{df(x)}{dx}\right]\left[\frac{dx(t)}{dt}\right]\\ &= [2x]\ \ [3kt^2]\\ &=[2(kt^3)]\,[3kt^2]\\ &=6k^2t^5.\end{aligned}$$

**i.)** Bottom Line: It is not correct to treat the symbolic representation of a derivative as though it were made up of algebraic symbols. Nevertheless, doing so has allowed us to derive a Chain Rule that looks and acts just like the one derived by mathematicians.

## D.) Partial Derivatives and the Del Operator:

**Note:** The *del operator* is a mathematical entity that most students first encounter at the university level. It is, nevertheless, a tool we will use later in this course. Of the material presented below, the *only* operations you will be expected to know will be the basic execution of a *partial derivative* and the basic execution of a *gradient*. Both will be discussed in class; neither will show up until much later in the course. The rest of the material has been included for the sake of completeness and for your own personal amusement.

**1.)** Consider a function $f(x,y)$ similar to the one graphed in Figure 2.5 to the right. Note that a *plane* parallel to the *y-z* plane has been placed at $x_1$, and that it cuts the

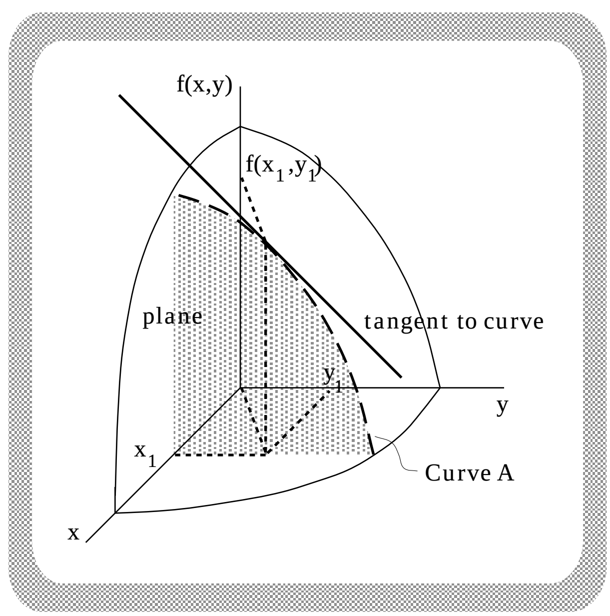

FIGURE 2.5

surface of *f(x,y)* along a curve as shown (the sketch labels this curve "Curve A").

**a.)** The question? How do we derive a function that tells us the *rate of change of f(x,y)* while holding the *x parameter* constant? Put another way, how do we mathematically define the *slope of the tangent to Curve A* in the sketch?

**b.)** In Figure 2.5, a tangent has been drawn on the curve at an arbitrary point $f(x_1, y_1)$. The slope of that tangent is:

$$\lim_{\Delta y \to 0} \frac{f(x_1, y_1 + \Delta y) - f(x_1, y_1)}{\Delta y}.$$

**c.)** This somewhat unusual operation (i.e., determining the slope of *f(x,y)* holding the *x* variable constant) is called a *partial derivative*. The symbol and formal definition for a partial derivative is shown below:

$$\frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x_1, y_1 + \Delta y) - f(x_1, y_1)}{\Delta y}.$$

**d.)** The *partial derivative* quoted above reads: the change of *f(x,y)* with respect to *y* holding *x* constant. Put another way, it is the derivative of *f(x,y)* with respect to *y* treating all other variables as constants.

**e.)** Example: $\partial(x^2y^4)/\partial y = (x^2)(4y^3)$.

**2.)** Partial derivatives are interesting in and of themselves, but they become really useful when used in the context of what is called the *del operator*.

**a.)** The *del operator* is defined as:

$$\nabla = \left[\frac{\partial()}{\partial x}\mathbf{i} + \frac{\partial()}{\partial y}\mathbf{j} + \frac{\partial()}{\partial z}\mathbf{k}\right].$$

**Note 1:** Bold-face letters denote vectors. The bold-face letters **i**, **j**, and **k** denote *unit vectors* in the *x*, *y*, and *z* directions respectively. As such, the *del operator* is a *vector operator*.

**Note 2:** An operator has no significance by itself. That is, it must operate on a function, be it scalar or vector, to derive any meaning.

**Note 3:** The "()'s" found in the *del operator* expression normally house the function being operated upon. Examples are coming.

**3.)** The *del operator* and the GRADIENT of a scalar:

**a.)** A *scalar field* is a function that defines a magnitude at every point in a given volume. *Temperature* is a *scalar field* as there exists a value for it everywhere.

**b.)** Consider the temperature *scalar field* $T = 3ky^2 + 20$ (this could model a room in which the temperature increases as one moves upward away from the floor). Assume that $k = 1\ ^o/m^2$ (it is included for the sake of units).

**c.)** Note that if we travel in the $+x$ direction, the temperature stays the same. Only when we move upward or downward does it change. In fact, the MAXIMUM POSITIVE CHANGE of $T$ is along the "$+\mathbf{j}$" direction.

**d.)** Using the *del operator* in conjunction with $T$, we get:

$$\nabla T = \left[\frac{\partial(3ky^2+20)}{\partial x}\mathrm{i} + \frac{\partial(3ky^2+20)}{\partial y}\mathrm{j} + \frac{\partial(3ky^2+20)}{\partial z}\mathrm{k}\right]$$
$$= (6ky)\mathrm{j}.$$

**e.)** Two things to observe:

**i.)** The DIRECTION of $\boldsymbol{\nabla} T$ is $+\boldsymbol{j}$. This is the same direction as the MAXIMUM POSITIVE CHANGE OF $T$ at any point.

**ii.)** Though it may not be immediately evident, the RATE OF CHANGE OF THE FUNCTION IN THE DIRECTION OF THE MAXIMUM POSITIVE CHANGE at any point is equal to the MAGNITUDE of $\boldsymbol{\nabla} T$ evaluated at that point.

**f.)** Bottom Line: When a *del operator* acts on a scalar function, it yields a vector function. The *direction* of the vector is THE DIRECTION IN WHICH THE SCALAR FUNCTION CHANGES THE FASTEST (in a positive sense), and the *magnitude* of the vector equals the RATE AT WHICH THE SCALAR FUNCTION CHANGES IN THAT FASTEST-CHANGING DIRECTION.

**g.)** This operation is called *the GRADIENT*.

**NOTE: THE REST OF THIS CHAPTER IS FOR YOUR OWN EDIFICATION. READ IT BUT DO NOT STRESS OVER IT!**

**4.)** The *del operator* and the DIVERGENCE of a vector:

**a.)** A *vector field* is a *vector function* that defines a magnitude and a direction at every point in a given volume. As an example, the earth's *gravitational field* is a vector field in the sense that at every point around the earth: **i.)** there is a number that defines the *magnitude* of the acceleration an object will experience if released at that point, and **ii.)** there is a *direction* defined at every point for that acceleration.

**b.)** Consider a cylindrical cavity that is closed at both ends and that holds air at high pressure (see Figure 2.6a).

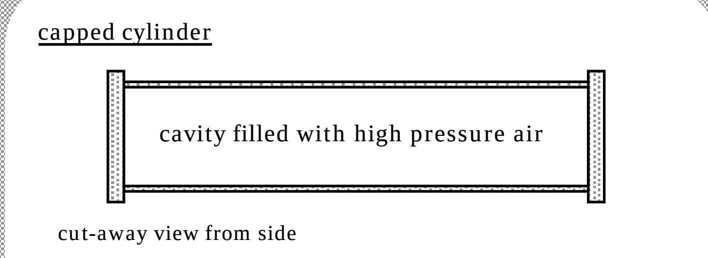


FIGURE 2.6a

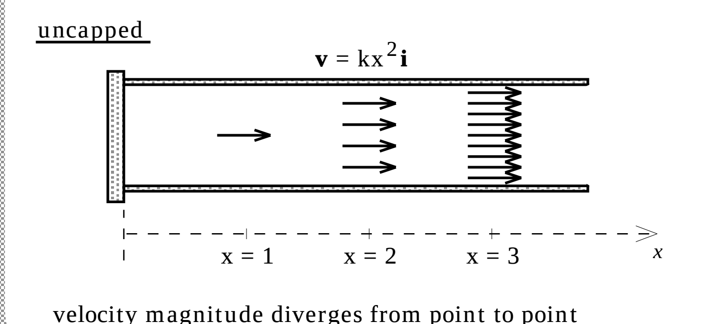


FIGURE 2.6b

**Note:** This example was inspired by a problem in Hugh Skilling's book *Fundamentals of Electric Waves.*

**i.)** When the cavity is opened, the compressed air will expand out and rush from the cavity in the process.

**ii.)** Just after the cavity is opened, the air's velocity near the *closed end* will be relatively small while the air's velocity near the *open end* will be large. For the sake of argument, assume the spatially varying velocity is defined as $\boldsymbol{v} = kx^2\boldsymbol{i}$, where $k$ is a constant with the appropriate units and $x$ is defined from the closed end (see Figure 2.6b).

**iii.)** The air's velocity is clearly diverging from point to point.

**iv.)** If we dot the *del operator* into the *velocity vector*, we get:

$$\nabla \bullet v = \left[\frac{\partial()}{\partial x} i + \frac{\partial()}{\partial y} j + \frac{\partial()}{\partial z} k\right] \bullet (kx^2 i + 0j + 0k)$$
$$= \left[\frac{\partial(kx^2)}{\partial x}\right]$$
$$= 2kx.$$

**v.)** This scalar quantity tells us the rate at which the velocity changes as we move along the line of the velocity vector. Put another way, it gives us the divergence of the velocity--*the rate of change of velocity with position*. This is *the DIVERGENCE* of the velocity vector.

**vi.)** In general, the DIVERGENCE yields *the rate of change of a vector function (in the direction of the vector) with position*.

**Note:** Although the above example is that of a diverging velocity, the divergence is most often associated with the *differential form of Gauss's Law* as it pertains to *electric fields* generated by *symmetric, extended charge* configurations. Most Calculus based physics texts (Halliday and Resnick's *Fundamentals of Physics*, for instance) do not present this form of Gauss's Law (they only give Gauss's Law in *INTEGRAL form*), but the *DIFFERENTIAL form* presented below is useful as well as being the *form of choice* in most advanced *electricity and magnetism* books. It has been included here because it is a practical example of the use of the divergence operation, and because it allows the student a peek at a relatively sophisticated mathematical tool physicists have at their disposal.

**c.)** In the world of static electricity, the evaluation of the *divergence* of an electric field ***E*** at a point is found to be proportional to the *charge density r* at the point. Mathematically, this is written:

$$\nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon_o},$$

where the symbol $\varepsilon_o$ is a constant. Called the *differential form* of *Gauss's Law*, this is one of Maxwell's equations.

**Note:** Again, when we get to Gauss's Law later in the year, we will deal with it in its *integral form*. Even so, the *differential* version definitely has its use.

**5.)** The *del operator* and the CURL of a vector:

**Note:** This operation is useful when dealing with the *differential form of Maxwell's equations* as they pertain to *magnetic fields*. As before, YOU WILL NOT BE TESTED ON THIS CONCEPT.

**a.)** Consider the force field $\boldsymbol{F} = (-bx^2)\boldsymbol{j}$ shown in Figure 2.7 to the right.

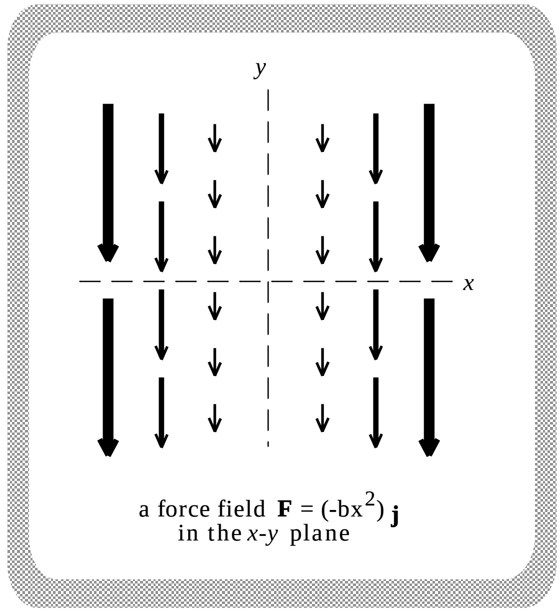

FIGURE 2.7

**b.)** Assume $b = 1$ *newton/meter* (i.e., it is there solely for units conformity). Both the *del operator* and ***F*** are vectors, so let's do a *cross product* between the two and see what we get:

$$\nabla x F = \begin{vmatrix} i & j & k \\ \frac{\partial()}{\partial x} & \frac{\partial()}{\partial y} & \frac{\partial()}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial()}{\partial x} & \frac{\partial()}{\partial y} & \frac{\partial()}{\partial z} \\ 0 & -bx^2 & 0 \end{vmatrix}$$

$$= i\left[\frac{\partial(0)}{\partial y} - \frac{\partial(-bx^2)}{\partial z}\right] + j\left[\frac{\partial(0)}{\partial z} - \frac{\partial(0)}{\partial x}\right] + k\left[\frac{\partial(-bx^2)}{\partial x} - \frac{\partial(0)}{\partial y}\right]$$
$$= (-2bx)k.$$

**c.)** Physical interpretation: Assume our *field* models the force of water flowing in a stream. That is, assume there is something in the middle of a stream which makes the flow negligible down the middle while allowing the water to flow faster as one proceeds out away from the central axis (we are obviously ignoring the fact that the stream's flow will slow down at the stream's shoreline). With this in mind, consider:

**i.)** Three paddle-wheels are placed in the field (see Figure 2.8)--one at *Point A*, one at *Point B*, and one at *Point C*. Each is positioned so that its *axis* is *perpendicular* to the stream's surface (i.e., in the *z direction*).

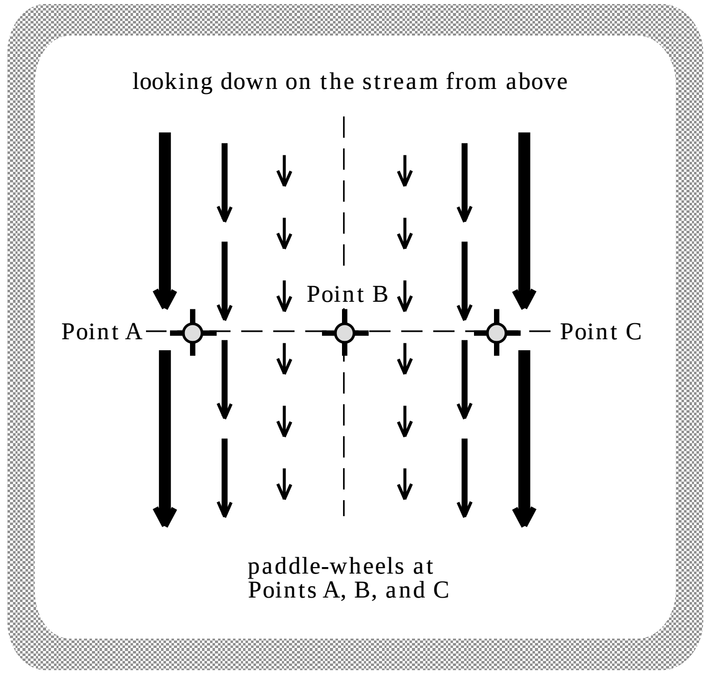


FIGURE 2.8

**ii.)** Due to the symmetry of the situation, there is no net force-of-rotation (i.e., no torque) on the paddle-wheel at *Point B*. That is, there is as much force pushing on the paddle to the left of the origin as there is on the paddle to the right of the origin.

Note also that at this point, the *CURL* of the force field $\nabla \times \mathbf{F} = (-2bx)\mathbf{k}$ is zero as $x = 0$.

**iii.)** Due to the asymmetry of the situation, there is a net torque acting on the paddle-wheel at *Point A* that makes it turn counterclockwise.

**Note:** The sign convention presented in the last chapter for the *direction* of a torque is as follows: A torque's UNIT VECTOR defines *the direction of the axis about which the rotation occurs:* the SIGN of the torque's *unit vector* defines the sense of the rotation (i.e., a *negative* sign implies clockwise rotation whereas a *positive* sign implies counterclockwise rotation).

**iv.)** Using our sign convention for torque, the stream-produced torque at *Point A* will, by inspection, be in the $+\mathbf{k}$ *direction* (the paddle-wheel will rotate counterclockwise).

**v.)** The *curl of* $\boldsymbol{F}$ (i.e., $\nabla \times \boldsymbol{F} = (-2bx)\boldsymbol{k}$) has a direction of $+\boldsymbol{k}$ for all negative values of $x$ (including that at *Point A*). In other words, the *curl* of $\boldsymbol{F}$ at *Point A* has the same sign and axis direction as the torque on the paddle-wheel at *Point A*.

**vi.)** A similar analysis for *positive* $x$ (i.e., for the paddle-wheel at *Point C*) yields a *curl* direction equal to $-\boldsymbol{k}$. This is the same direction as that of the paddle-wheel torque in *that* situation.

**vii.)** Lastly, although it is probably not as obvious, the MAGNITUDE of the *curl* yields the *rate* at which a paddle wheel will rotate at a given point in the field.

**d.)** In summary:

**i.)** The MAGNITUDE of the *curl* of a vector yields the *rate at which the field circulates* at a given point in the field.

**ii.)** The DIRECTION of the *curl* of a vector at a particular point tells you the *axis* about which the field *circulates* at that point. It also denotes the sense of the rotation (i.e., clockwise or counterclockwise).

**Note:** The word *circulate* is being used loosely here. Its significance depends upon the system in question. In our example, it was associated with the direction of the torque provided by the stream on a paddle-wheel. We could as easily have been dealing with electrical phenomena. For example, a magnetic field $\boldsymbol{B}$ that *changes with time* induces an electric field $\boldsymbol{E}$ that circles. The rate at which the magnetic field changes $\partial \boldsymbol{B} / \partial t$ is related to the *curl of* $\boldsymbol{E}$ such that:

$$\nabla \mathrm{x} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

The moral of the story? How one interprets the word "circulation" is driven by the context of the problem.

# QUESTIONS

There are none! This chapter was not intended to be an exercise in skills-learning (hence, no need for questions to test your understanding). It was designed as a discussion aimed at giving you a feel for how Calculus works and, more to the point, how physicists treat Calculus in the analysis of *real-world* problems. If the material has made sense, great. If not, don't worry about it. You will have plenty of time to become acquainted with Calculus, physics style, as we go.